Learning to Branch in Mixed Integer Programming

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AAAI 2016

Joint work with Pierre Le Bodic, Le Song,
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Overview

- Goal: solve Mixed Integer Linear Programs (MIP) "faster"
- Branch-and-Bound (B&B): a general framework for solving MIPs
- B&B components: presolve, cutting planes, primal heuristics, **branching** (this talk)

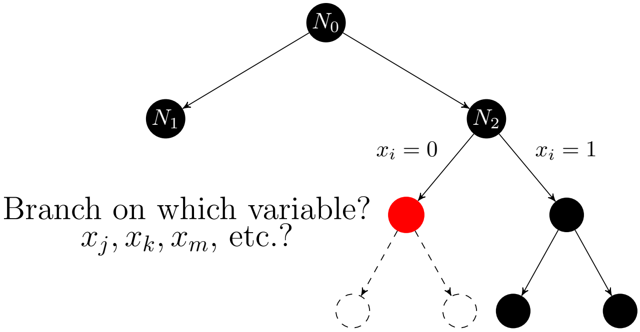

Figure: A B&B tree. A node is a sub-problem, an edge is an additional constraint

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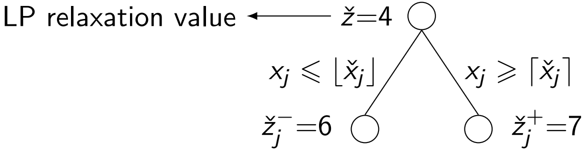
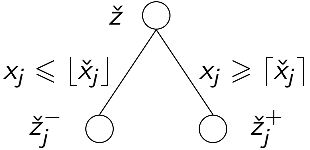


  - Branching on x_j prunes off the child sub-trees

Quality of a Branch

Measuring the quality of a branch:



- Change in objective value: $\Delta_j^- = \check{z}_j^- - \check{z}$ and $\Delta_j^+ = \check{z}_j^+ - \check{z}$
- Quality of the branch:

$$\text{score}(\Delta_j^-, \Delta_j^+) = \Delta_j^- \cdot \Delta_j^+$$

- Higher score is better

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- Strong Branching (SB)

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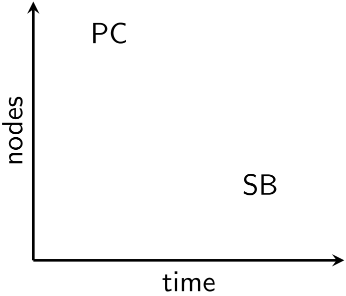

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- This talk: *can we get the best of both worlds (SB and PC)?*
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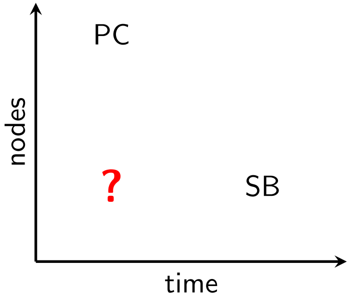

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Proposed Method

A Machine Learning (ML) framework for branching that imitates SB well, at a fraction of the computation cost

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  - Tempted to set up learning task as fitting a regression model to estimate SB scores directly
- Key observation:
  - SB scores are only used to **select** the *single* best variable
- Goal: learn to imitate the correct **ranking** of the candidate variables at each node
  - Collect training data on SB ranking & fit a ranking model

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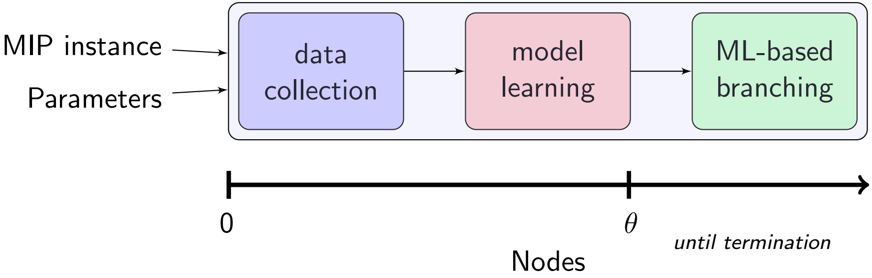
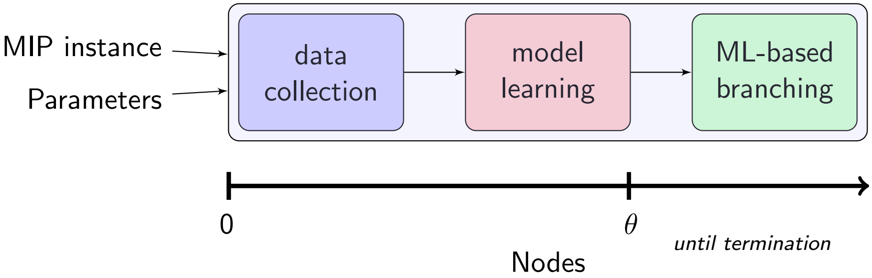


- **On-the-fly**: no upfront offline learning required
- **Instance-specific**: different ML model for each instance
- **No lost work**: data collection is part of the search tree

On-the-fly Learning of Branching Strategies



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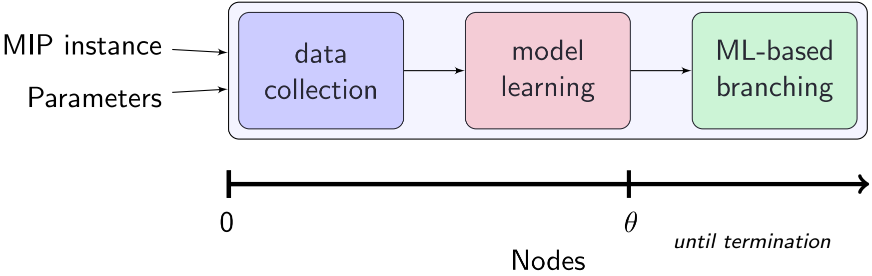

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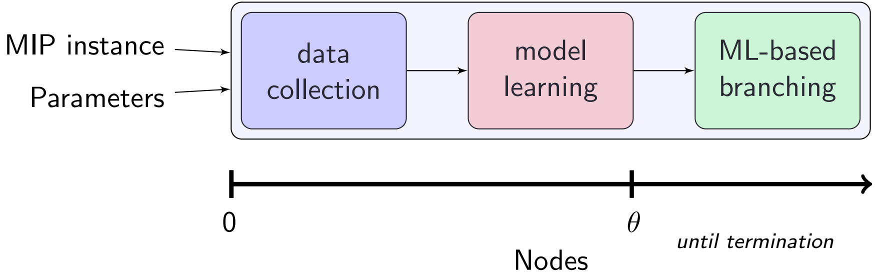

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ML-based branching: as of the $(\theta + 1)^{st}$ node, use learned function f to rank variables based on their features

Data Collection

For the first θ nodes, $N_i \in \mathcal{N} = \{N_1, ..., N_\theta\}$, run SB on some fractional integer variables $\mathcal{C}_i$, $|\mathcal{C}_i| \leqslant \kappa$, and collect dataset $\mathcal{D}$ formed of:

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Example dataset: $\theta = 2, \kappa = 4$, $p = 4$ features

		y_j^i	feature 1	feature 2	feature 3	feature 4
N_1	x_2	0	0.4	0.4	0.3	0.9
	x_4	1	0.3	0.9	0.4	0.9
	x_5	0	0.2	0.2	0.6	0.7
	x_6	1	0.5	0.8	0.4	0.4
N_2	x_1	1	0.2	0.2	0.6	0.9
	x_2	0	0.5	0.2	0.8	0.7
	x_4	0	0.4	0.9	0.1	0.4

Features

- Feature map $\phi : \mathcal{X} \times \mathcal{N} \rightarrow [0, 1]^p$, where $\mathcal{X} = \{x_1, ..., x_n\}$.
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- Scale and normalize all features, *per node*, to $[0,1]$.

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Static Features (18)		
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Dynamic Features (54)		
Slack and ceil distances	2	
Pseudocosts	5	[Achterberg, 2009]
Infeasibility statistics	4	
Statistics for constraint degrees	7	
Min/max for ratios of constraint coefficients to RHS	4	[Alvarez et al., 2014]
Min/max for one-to-all coefficient ratios	8	[Alvarez et al., 2014]
Stats. for active constraint coefficients	24	[Patel and Chinneck, 2007]

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- **Relaxed Binary Labels**:

$$y_j^i = \begin{cases} 1, & \text{if SB score of } x_j \text{ is within } \alpha \text{ of max. SB score at } N_i \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha \in (0, 1)$ is small.

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- Open-source implementation SVM^{rank}: `http://www.cs.cornell.edu/people/tj/svm_light/svm_rank.html`

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- Complexity (per node):
  - $O\big(nnz(A)\big)$ for feature computation (line 2)
  - $O(p \cdot \kappa)$ for scoring by dot product (line 3)

Algorithm: ML-based Branching

Input: node N_i, candidate set $\mathcal{C}_i, |\mathcal{C}_i| \leqslant \kappa$
Output: variable x_k s.t. $k \in \mathcal{C}_i$

```
1 for each j ∈ C_i do
2     Compute the features by φ(x_j, N_i)
3     Compute the score s_j = f(φ(x_j, N_i))
4 end
5 Branch on x_k, where s_k = max_{j∈C_i} {s_j}
```

Experimental Setup

- All algorithms implemented using control callbacks in CPLEX 12.6.1
- Cuts applied at the root, disabled afterwards
- Optimal value provided as upper cutoff
- Number of simplex iterations for any SB call is 50
- MIPLIB2010 "Benchmark" set with a time cutoff of 5 hours; default gap tolerances

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- MIPLIB2010 "Benchmark" set; 84 feasible instances, each with 10 random seeds. From the 840 instances, 523 remain after filtering (too easy, too hard, etc.)
- Similar experimental procedures used in [Fischetti and Monaci, 2012, Kılınç-Karzan et al., 2009].

Results

		CPLEX-D	SB	PC	SB+PC	SB+ML
Unsolved Instances	All (523)	11	129	66	63	**52**
	Easy (255)	0	12	15	14	**13**
	Medium (120)	2	43	22	22	**17**
	Hard (148)	9	74	29	27	**22**

- An instance is "Easy" if CPLEX-D solves in $\leqslant$ 50,000 nodes, "Medium" in $\leqslant$ 500,000 nodes, "Hard" otherwise
- PC, SB+PC and SB+ML are competing, lower is better for all three criteria, and best is in bold

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Num. nodes	All (523)	46,633	33,072	92,662	70,455	**59,223**
	Easy (255)	3,255	3,610	7,931	5,224	**5,124**
	Medium (120)	173,417	121,923	395,199	288,916	**234,093**
	Hard (148)	1,570,891	519,878	1,971,333	1,979,660	**1,314,263**

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Total time	All (523)	499	2,263	**960**	1,093	1,059
	Easy (255)	111	602	**243**	361	382
	Medium (120)	1,123	6,169	2,493	1,892	**1,776**
	Hard (148)	3,421	9,803	4,705	4,718	**4,039**

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Head-to-head Comparisons: Win/Tie/Loss

	CPLEX-D	SB	PC	SB+PC
CPLEX-D				
SB	5/264/0/125/123			
PC	8/164/0/285/63	68/63/0/326/5		
SB+PC	8/227/0/225/60	72/66/7/315/6	15/320/0/125/12	
SB+ML	8/267/0/196/49	82/96/7/286/5	21/355/0/95/7	17/300/58/96/6

Table: Each cell has the numbers of: Abs. Win/Win/Tie/Loss/Abs. Loss

Compare branching strategy $\mathcal{A}$ to $\mathcal{B}$:

- *absolute win* for $\mathcal{A}$ vs. $\mathcal{B}$ on instance $\mathcal{I} \Leftrightarrow \mathcal{A}$ solves $\mathcal{I}$, $\mathcal{B}$ does not
- *win* for $\mathcal{A}$ vs. $\mathcal{B}$ on $\mathcal{I} \Leftrightarrow \mathcal{A}$ and $\mathcal{B}$ solve $\mathcal{I}$, and $\mathcal{A}$ does so in fewer nodes than $\mathcal{B}$
- *tie* between $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{I} \Leftrightarrow \mathcal{A}$ and $\mathcal{B}$ solve $\mathcal{I}$ in same num. of nodes

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- *absolute win* for $\mathcal{A}$ vs. $\mathcal{B}$ on instance $\mathcal{I} \Leftrightarrow \mathcal{A}$ solves $\mathcal{I}$, $\mathcal{B}$ does not
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Head-to-head Comparisons: Win/Tie/Loss

	CPLEX-D	SB	PC	SB+PC
CPLEX-D				
SB	5/264/0/125/123			
PC	8/164/0/285/63	68/63/0/326/5		
SB+PC	8/227/0/225/60	72/66/7/315/6	15/320/0/125/12	
SB+ML	8/267/0/196/49	82/96/7/286/5	21/355/0/95/7	17/300/58/96/6

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Future Directions

- Adaptive learning w.r.t. the evolution of the search

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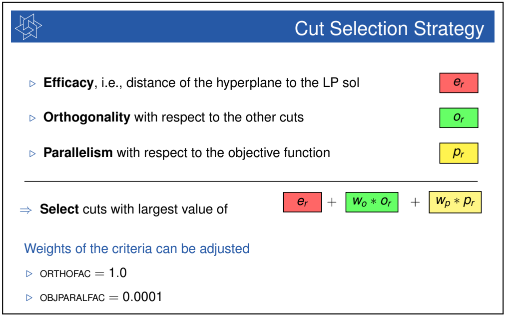
- Adaptive learning w.r.t. the evolution of the search
- Fully online or offline learning
- Most decisions in the MIP solver are inherently heuristic
  - Can we use data and ML to make better informed decisions?

Figure: Cut Selection Strategy in SCIP (slide from [Wolter, 2006])



The End

Thanks!
Questions?

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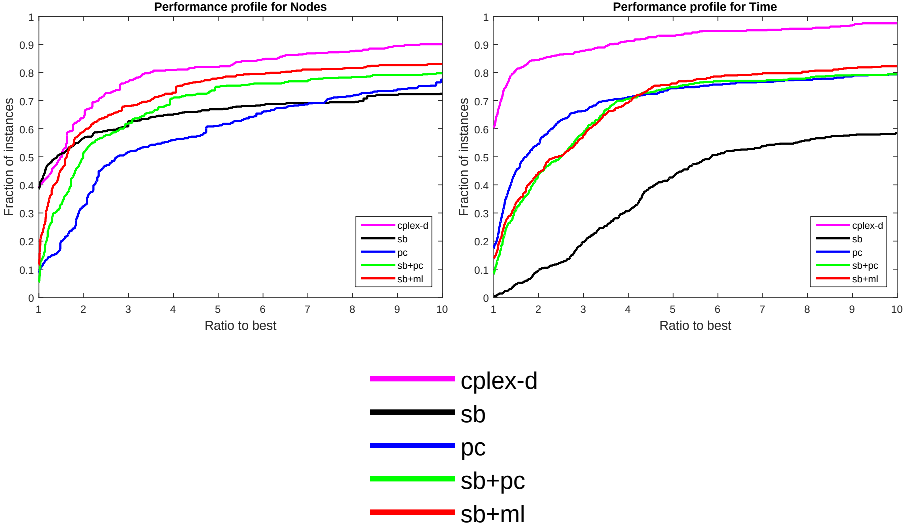
Features Description

Feature	Description	Count	Reference
Static Features (18)			
Objective function coeffs.	Value of the coefficient (raw, positive only, negative only)	3	
Num. constraints	Number of constraints that the variable participates in (with a non-zero coefficient)	1	
Stats. for constraint degrees	The *degree of a constraint* is the number of variables that participate in it. A variable may participate in multiple constraints, and statistics over those constraints' degrees are used. The constraint degree is computed on the root LP (mean, stdev., min, max)	4	
Stats. for constraint coeffs.	A variable's positive (negative) coefficients in the constraints it participates in (count, mean, stdev., min, max)	10	
Dynamic Features (54)			
Slack and ceil distances	$\min\{\breve{x}^i_j - \lfloor \breve{x}^i_j \rfloor, \lceil \breve{x}^i_j \rceil - \breve{x}^i_j\}$ and $\lceil \breve{x}^i_j \rceil - \breve{x}^i_j$	2	
Pseudocosts	Upwards and downwards values, and their corresponding ratio, sum and product, weighted by the fractionality of x_j	5	(Achterberg 2009)
Infeasibility statistics	Number and fraction of nodes for which applying SB to variable x_j led to one (two) infeasible children (during data collection)	4	
Stats. for constraint degrees	A dynamic variant of the static version above. Here, the constraint degrees are on the current node's LP. The ratios of the static mean, maximum and minimum to their dynamic counterparts are also features	7	
Min/max for ratios of constraint coeffs. to RHS	Minimum and maximum ratios across positive and negative right-hand-sides (RHS)	4	(Alvarez, Louveaux, and Wehenkel 2014)
Min/max for one-to-all coefficient ratios	The statistics are over the ratios of a variable's coefficient, to the sum over all other variables' coefficients, for a given constraint. Four versions of these ratios are considered: positive (negative) coefficient to sum of positive (negative) coefficients	8	(Alvarez, Louveaux, and Wehenkel 2014)
Stats. for active constraint coefficients	An active constraint at a node LP is one which is binding with equality at the optimum. We consider 4 weighting schemes for an active constraint: unit weight, inverse of the sum of the coefficients of all variables in constraint, inverse of the sum of the coefficients of only candidate variables in constraint, dual cost of the constraint. Given the absolute value of the coefficients of x_j in the active constraints, we compute the sum, mean, stdev., max. and min. of those values, for each of the weighting schemes. We also compute the weighted number of active constraints that x_j is in, with the same 4 weightings	24	(Patel and Chinneck 2007)

Performance Profiles



Head-to-head Comparisons: Node Ratios

	CPLEX-D	SB	PC	SB+PC	SB+ML
CPLEX-D		1.39 (389)	0.64 (449)	0.84 (452)	0.97 (463)
SB	0.72 (389)		0.47 (389)	0.61 (388)	0.76 (389)
PC	1.56 (449)	2.11 (389)		1.34 (445)	1.59 (450)
SB+PC	1.20 (452)	1.63 (388)	0.75 (445)		1.22 (454)
SB+ML	1.03 (463)	1.32 (389)	0.63 (450)	0.82 (454)	

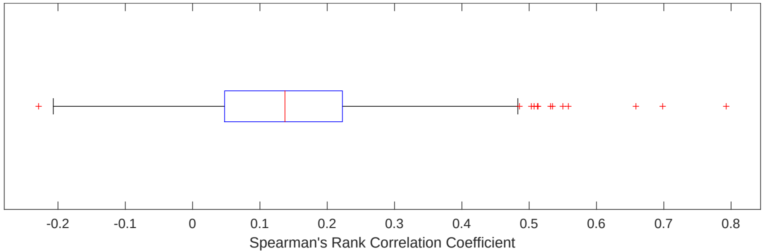
Head-to-head Comparisons: Win/Tie/Loss (averaging over seeds)

	CPLEX-D	SB	PC	SB+PC
CPLEX-D				
SB	45/0/20			
PC	20/0/45	5/0/60		
SB+PC	28/0/37	8/0/57	51/0/14	
SB+ML	33/0/32	10/0/55	55/0/10	46/6/13

Table: Win-tie-loss counts for every pair of strategies, for the number of nodes. A triplet in a cell in row $\mathcal{A}$ and column $\mathcal{B}$ expresses the number of wins for $\mathcal{A}$ over $\mathcal{B}$, the number of ties, and the number of losses of $\mathcal{A}$ to $\mathcal{B}$, respectively, w.r.t. to the shifted geometric mean of the number of nodes. $\mathcal{A}$ wins over $\mathcal{B}$ on instance $\mathcal{I}$ iff the mean number of nodes of $\mathcal{A}$ over the random seeds on $\mathcal{I}$ is strictly less than that of $\mathcal{B}$; losses and ties are defined analogously.

Analysis of Feature Ranking Models

- Compare ranking models across instances
- For each model, consider the ranking of the features by their absolute weight in the linear model
- Similarity of two models measured by *Spearman's rank correlation coefficient* between their respective feature weight rankings



Analysis of Feature Ranking Models

- The 10 features that appear the most in the top $K = 10$ features over models, sorted by that count (corresponding column is bold in the header). The counts for other values of K are also shown. There are 2700 features in total. Static features are marked by (S); unmarked features are dynamic.

Feature	1	3	5	**10**	20	100
PC Product	1	10	17	23	30	37
PC Product x PC Product	6	10	11	17	24	38
PC Product x Mean Constraint Degree	1	2	5	13	19	33
PC Product x Max. Constraint Degree	0	0	3	13	19	34
PC Product x Min. Constraint Degree	0	3	6	13	21	34
PC Product x Max. Absolute Coefficient in Active Constraints	2	7	8	13	27	37
PC Product x Mean Absolute Coefficient in Active Constraints	2	4	4	10	25	35
PC Product x Num. Constraints (S)	0	3	5	9	17	28
PC Product x Min. Positive Constraint Coefficient (S)	1	1	2	9	18	32
PC Product x Max. Ratio of Constraint Degree (static/dynamic)	0	3	4	9	16	34

Results **with cuts and heuristics, no upper cutoff**

- "Easy" if CPLEX solves in $\leqslant$ 50,000 nodes, "Medium" in $\leqslant$ 500,000 nodes, "Hard" otherwise
- PC, SB+PC and SB+ML are competing, lower is better for all three criteria, and best is in bold

		CPLEX-D	SB	PC	SB+PC	SB+ML
Instances unsolved	All (538)	28	139	55	59	**54**
	Easy (318)	3	23	21	22	**20**
	Medium (140)	18	74	**24**	26	27
	Hard (80)	**7**	42	10	11	**7**
Num. nodes	All (538)	25,407	17,909	43,120	36,235	**32,948**
	Easy (318)	3,643	3,794	7,371	5,386	**5,358**
	Medium (140)	165,188	91,279	260,851	266,497	**217,306**
	Hard (80)	2,144,179	491,557	2,062,439	2,142,617	**1,649,930**
Total time	All (538)	681	2,273	**1,027**	1,229	1,423
	Easy (318)	203	821	**385**	498	630
	Medium (140)	2,978	9,942	**3,767**	4,077	4,210
	Hard (80)	6,223	9,815	**5,192**	5,461	5,423

Head-to-head Comparisons: Win/Tie/Loss

	CPLEX-D	SB	PC	SB+PC
CPLEX-D				
SB	5/259/0/135/116			
PC	20/183/0/280/47	98/60/0/325/14		
SB+PC	21/216/0/242/52	94/96/1/288/14	22/264/0/193/26	
SB+ML	22/231/0/231/48	98/108/1/277/13	23/313/0/148/22	24/260/55/145/19

Definitions:

- *absolute win* for $\mathcal{A}$ v/s $\mathcal{B}$ on instance $\mathcal{I} \Leftrightarrow \mathcal{A}$ solves $\mathcal{I}$, $\mathcal{B}$ does not
- *win* for $\mathcal{A}$ v/s $\mathcal{B}$ on $\mathcal{I} \Leftrightarrow \mathcal{A}$ and $\mathcal{B}$ solve $\mathcal{I}$, and $\mathcal{A}$ does so in $<$ nodes than $\mathcal{B}$
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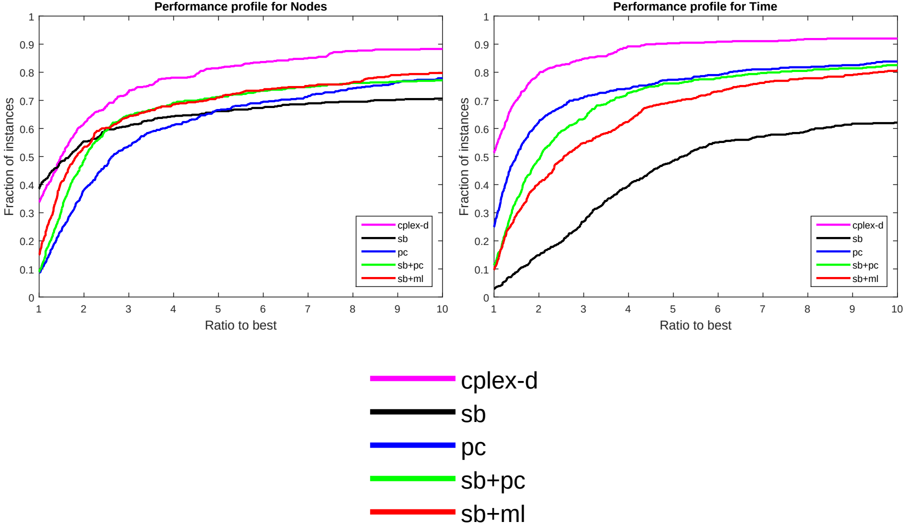
Head-to-head Comparisons: Node Ratios

	CPLEX-D	SB	PC	SB+PC	SB+ML
CPLEX-D		1.38 (394)	0.64 (463)	0.79 (458)	0.84 (462)
SB	0.72 (394)		0.48 (385)	0.65 (385)	0.67 (386)
PC	1.56 (463)	2.09 (385)		1.24 (457)	1.32 (461)
SB+PC	1.27 (458)	1.55 (385)	0.81 (457)		1.08 (460)
SB+ML	1.20 (462)	1.49 (386)	0.76 (461)	0.92 (460)	

Table: Ratios for the shifted geometric means (shift 10) over nodes on instances solved by both strategies. The first value in a cell in row $\mathcal{A}$ and column $\mathcal{B}$ is the ratio of the average number of nodes used by $\mathcal{A}$ to that of $\mathcal{B}$. The second value is the number of instances solved by both $\mathcal{A}$ and $\mathcal{B}$.

Performance Profiles



Pseudocost Branching (PC)

- Goal: Imitate SB without many simplex iterations.
- Record change in objective value over time, and use historical averages as proxy for SB scores.
- Objective increase per unit change in x_j at N_i when branching down:

$$\varsigma_{i,j}^{-} = \frac{\Delta_j^{-}}{f_j^{-}}$$

with $f_j^{-} = \check{x}_j^i - \lfloor \check{x}_j^i \rfloor$

- Take average over those events :

$$\Psi_j^{-} = \frac{\sum \varsigma_{i,j}^{-}}{\#\text{ times branched down on } x_j}$$

- PC score at some node:

$$s_j = \mathrm{score}\big(f_j^{-}\Psi_j^{-}, f_j^{+}\Psi_j^{+}\big)$$

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Wolter, K. (2006).
Cutting plane separators in SCIP.

Related Work

- [Achterberg and Berthold, 2009],[Hendel, 2015]: recent PC-based strategies
- [Kılınç-Karzan, Savelsbergh and Nemhauser, 2009]: collect information on which variables are likely to lead to fathomed child nodes quickly, then restart solve and branch on those
- [He et al., 2014]: ML (classification) for node selection; offline, can prune optimum
- [Alvarez et al., 2014]: ML (regression) for variable selection; offline, slow and modest results